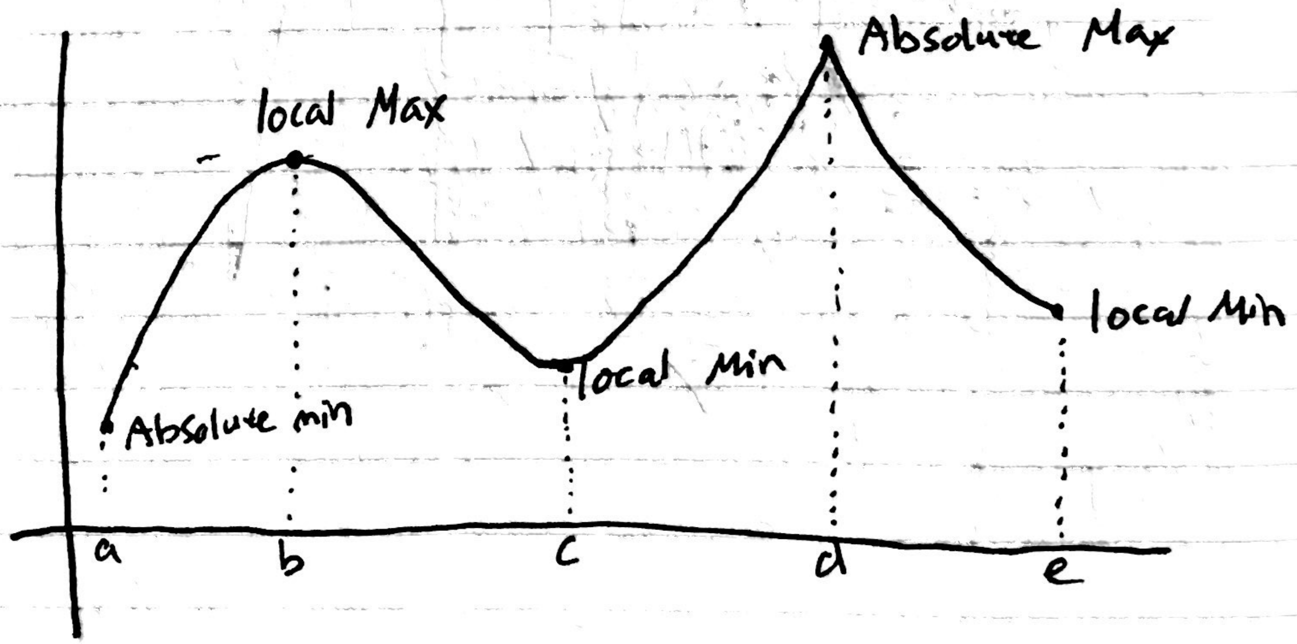
Chapter 3: Applications of Differentiation

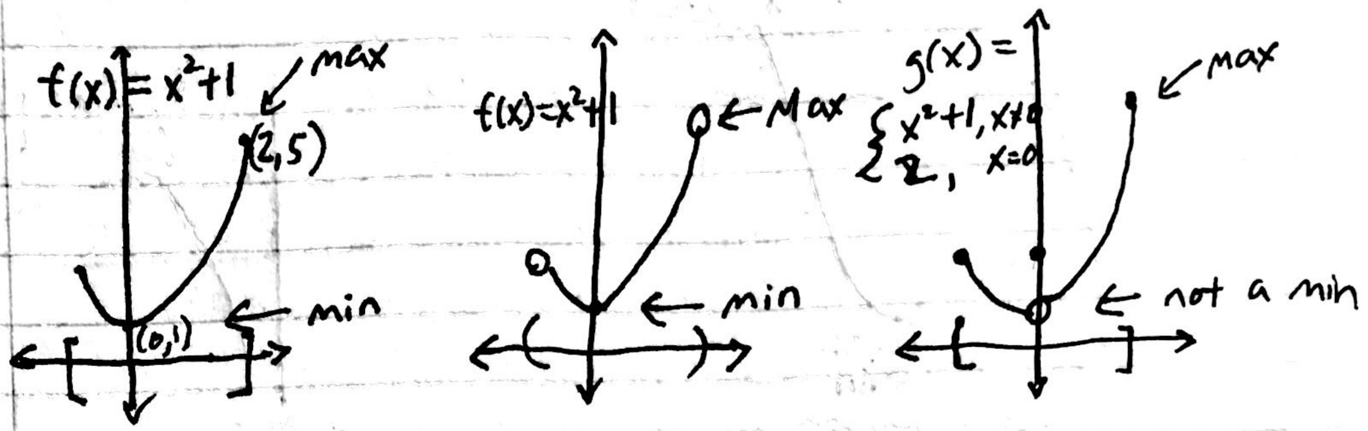
Definition of Extrema

Let f be determined on an interval I containing c.

1. $f(c)$ is the minimum of f on I if $f(c) \le f(x)$ for all x in I.
2. $f(c)$ is the maximum of f on I if $f(c) \ge f(x)$ for all x in I.

The minimum and maximum of a function on an interval are the extreme values, or extrema, of the function on the interval. The minimum and maximum of a function on an interval are also called the absolute minimum and absolute maximum, or global minimum and global maximum, on the interval





f is continuous on [-1, 2] is closed

f is continuous (-1, 2) is open

g is not continuous, [-1, 2] is closed. Extrema can occur at interior points or endpoints of an interval. Extrema that occur at endpoints are called endpoint extrema

Definition of Relative Extrema

1. If there is an open interval containing c on which f(c) is a maximum, then f(c) is called a relative maximum of f, or you can say that f has a relative maximum at (c, f(c))
2. If there is an open interval containing c on which f(c) is a minimum, then f(c) is called a relative minimum of f, or you can say that f has a relative minimum at (c, f(c))

Relative is AKA local

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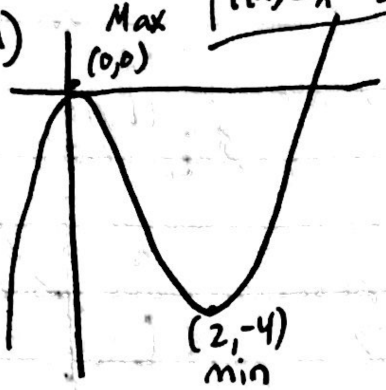
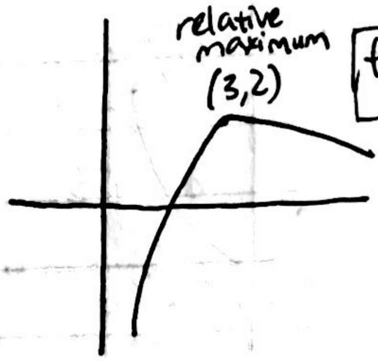
a) $\boxed{f(x)=x^3-3x^2}$



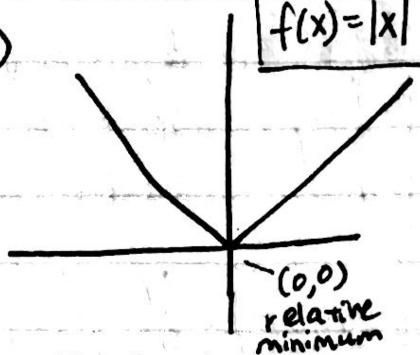
f has a relative maximum at (0,0) and a relative minimum at (2,-4)

b) $\boxed{f(x)=\frac{9(x^2-3)}{x^3}}$



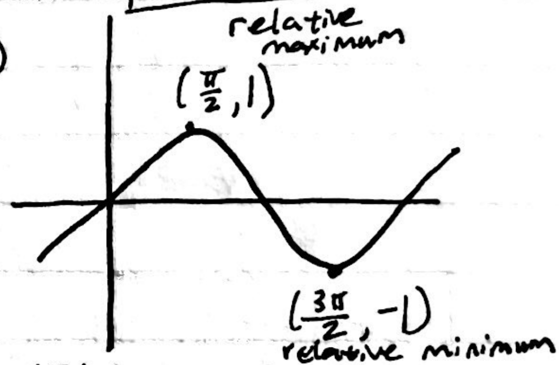
$f'(3)=0$

c) $\boxed{f(x)=|x|}$



$f'(0)$ does not exist

d) $\boxed{f(x)=\sin x}$

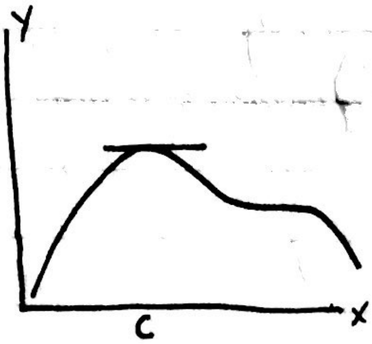


$f'(\frac{\pi}{2})=0$ $f'(\frac{3\pi}{2})=0$

The value of the derivative at the relative maximums and minimums is 0.

Relative Extrema occur only at Critical Numbers

definition of a Critical Number:
If $f'(c)=0$, c is a critical number of f



(c, f(c)) is relative maximum
$f'(c)=0$

Guidelines for finding Extrema on a Closed Interval
1. Find the critical numbers of f in (a,b)
2. Evaluate f at each critical number in (a,b)
3. Evaluate f at each endpoint of [a,b].
4. The least of these values is the minimum, The greatest is the maximum

Finding Extrema on a Closed Interval

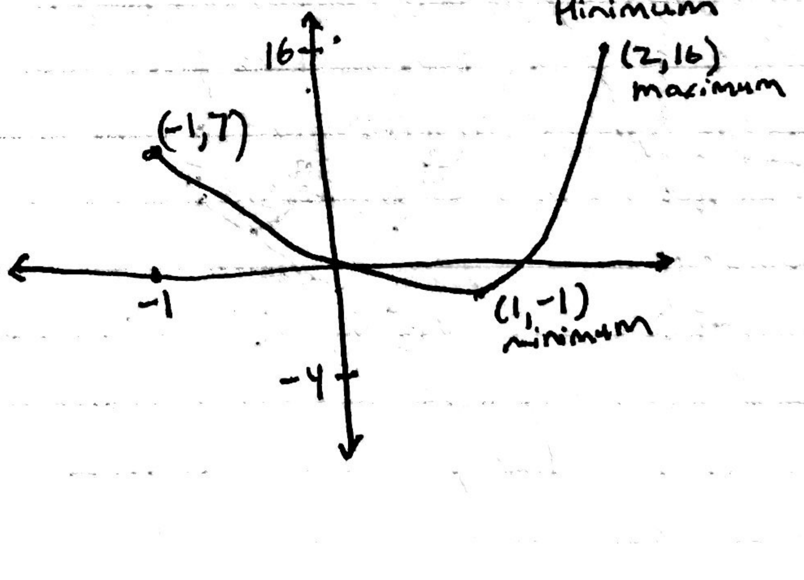
$f(x)=3x^4-4x^3 \quad [-1,2]$

$f'(x)=12x^3-12x^2=0$

$12x^2(x-1)=0$

$x=0,1$ ← these are critical numbers

Left Endpoint	Critical Number	Critical Number	Right Endpoint
$f(-1)=7$	$f(0)=0$	$f(1)=-1$	$f(2)=16$
		↑ Minimum	↑ maximum



Finding Extrema on a Closed Interval Example 2

given: $f(x) = 2\sin x - \cos 2x \quad [0, 2\pi]$

$f'(x) = 2\cos x + 2\sin 2x = 0$

$2(\cos x)(1 + 2\sin x) = 0$

$\cos x = 0$

$x = \frac{\pi}{2}, \frac{3\pi}{2}$

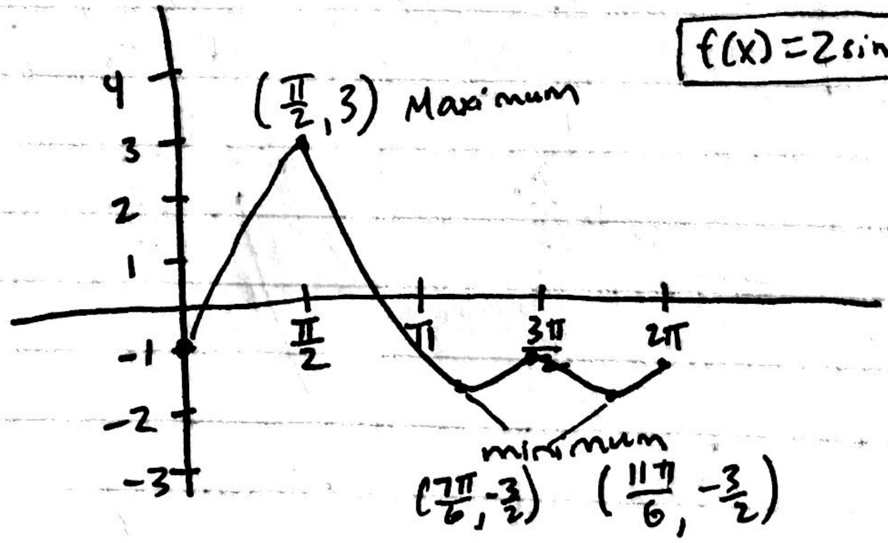
$1 + 2\sin x = 0$

$\sin x = -\frac{1}{2}$

$x = \frac{7\pi}{6}, \frac{11\pi}{6}$

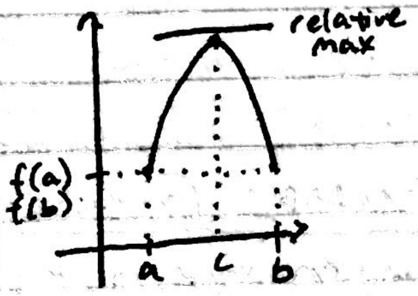
Left Endpoint	Critical Number	Critical Number	Critical Number	Critical Number	Right Endpoint
$f(0) = -1$	$f(\frac{\pi}{2}) = 3$	$f(\frac{7\pi}{6}) = -\frac{3}{2}$	$f(\frac{3\pi}{2}) = -1$	$f(\frac{11\pi}{6}) = -\frac{3}{2}$	$f(2\pi) = -1$
	Maximum	Minimum		Minimum	

$\boxed{f(x) = 2\sin x - \cos x}$

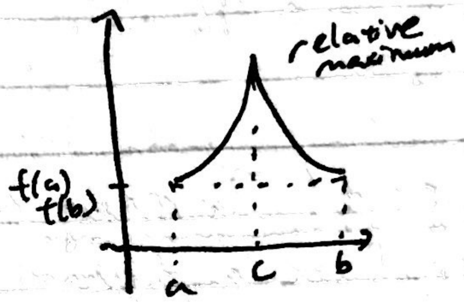


Rolle's Theorem

Let f be continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b). If $f(a)=f(b)$ then there is at least one number c in (a,b) Such that $f'(c)=0$



$f(a)=f(b)$ $f'(c)=0$
f is continuous on $[a,b]$ and differentiable on (a,b)

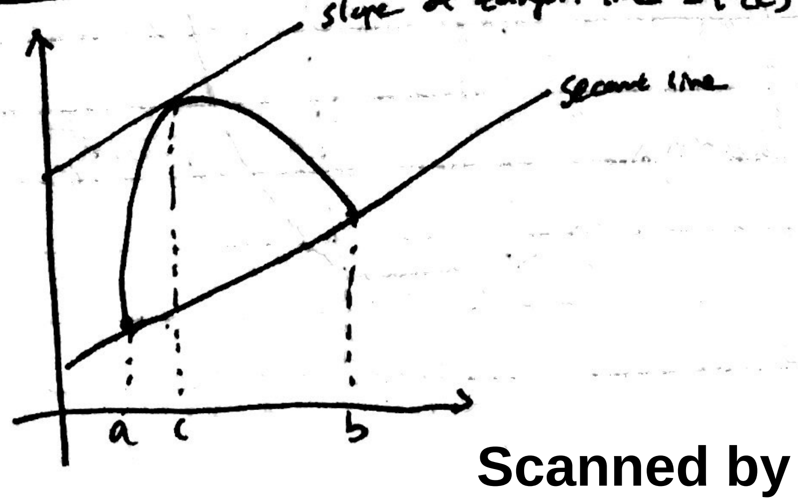


f is continuous on $[a,b]$ but is NOT differentiable on (a,b)
$f'(c)$ = cannot differentiate

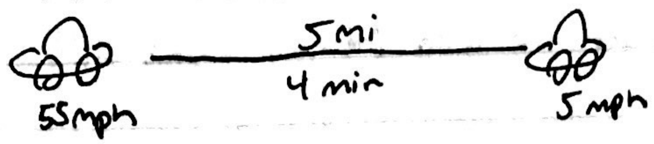
The Mean Value Theorem

If f is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b), then there exists a number c in (a,b) such that

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$



Finding the instantaneous rate of change



(t, d)

(0, 0)

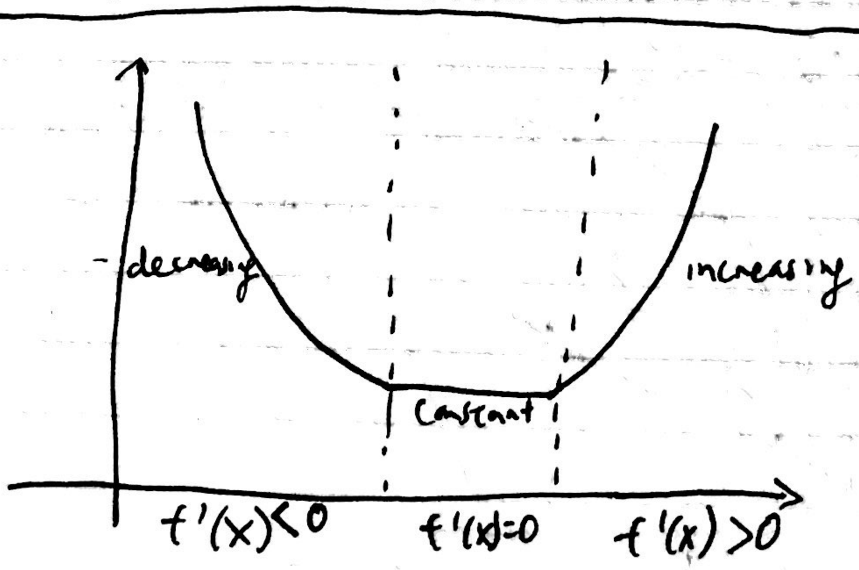
$(\frac{1}{15}$ hr, 5mi$)$

$$\frac{5-0}{\frac{1}{15}-0} = 75 \text{ mph}$$

Test for Increasing and Decreasing Functions (3.5)

Let f be a function that is continuous on the closed interval [a,b] and differentiable on the open interval (a,b)

1. If $f'(x) > 0$ for all x in (a,b), f is increasing on [a,b]
2. If $f'(x) < 0$ for all x in (a,b), f is decreasing on [a,b]
3. If $f'(x) = 0$ for all x in (a,b), f is constant on [a,b]



Guidelines for finding intervals on which a function is increasing or decreasing
Let f be continuous on the interval (a,b). To find the open intervals on which f is increasing or decreasing, use the following steps 1) Locate the critical #s of f in (a,b) and use these numbers to determine test intervals 2) Determine the sign of $f'(x)$ for at least one test value in each of the intervals 3). Use theorem 3.5 to determine whether f is increasing or decreasing on each interval

e.g. $f(x) = x^3 - \frac{3}{2}x^2$

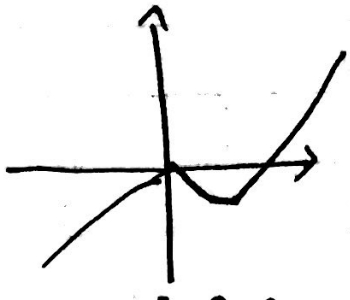
$f'(x) = 3x^2 - 3x$

$3x^2 - 3x = 0$

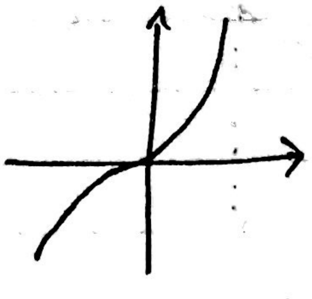
$3x(x-1) = 0$

$x=0 \quad x=1$ ← critical numbers

Interval	$(-\infty,0)$	$(0,1)$	$(1,\infty)$
test value	$x=-1$	$x=\frac{1}{2}$	$x=2$
$f'(x)$	$f'(-1)=6>0$	$f'(\frac{1}{2})=-\frac{3}{4}<0$	$f'(2)=6>0$
conclusion	increasing	decreasing	increasing



$f(x) = x^3 - \frac{3}{2}x^2$

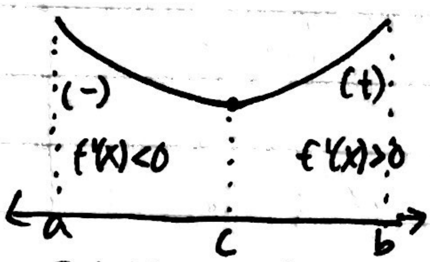


$f(x) = x^3$

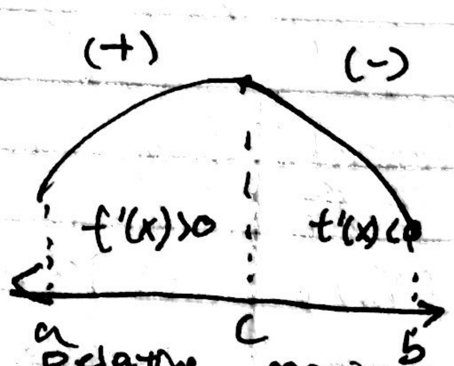
The First Derivative Test

Let c be a critical number of a function f that is continuous on an open interval I containing c. If f is differentiable on interval, except, possibly, at c, then $f(c)$ can be classified as follows

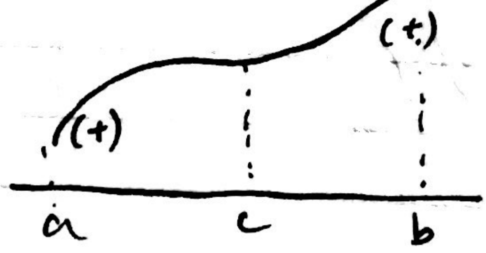
1) If $f'(x)$ changes from negative to positave at c, then f has a relative minimum at $(c, f(c))$
2) If $f'(x)$ changes from positave to negative at c, then f has a relative maximum at $(c, f(c))$
3) If $f'(x)$ is positave on both sides of c, or negative on both sides of c, then $f(c)$ is neither a relative minimum nor a maximum

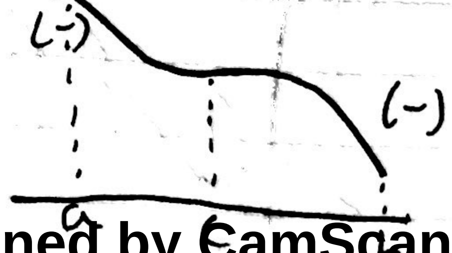


Relative minimum



Relative maximum





neither relative min nor max

example:

$$f(x) = x^3 - \frac{3}{2}x^2$$
$$f'(x) = 3x^2 - 3x$$
$$3x^2 - 3x = 0$$
$$3x(x-1) = 0$$
$$x = 0 \text{ or } 1$$ ← critical numbers

Interval	$(-\infty, 0)$	$(0, 1)$	$(1, \infty)$
Test value	-1	$\frac{1}{2}$	2
$f'(x)$	$+$	$-$	$+$
Conclusion	increasing	decreasing	increasing

⇒ 

$f(0) = 0$

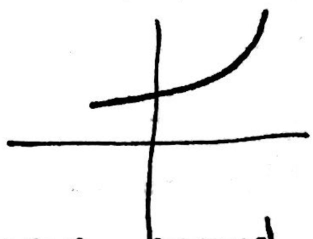
relative max at $(0,0)$

$f(1) = -\frac{1}{2}$

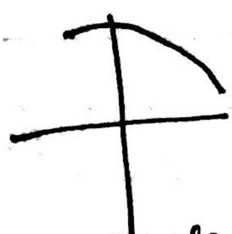
relative min at $(1, -\frac{1}{2})$

Definition of Concavity

Let f be differentiable on an open interval I. The graph of f is concave upward on I if f' is increasing on the interval and concave downward if f' is decreasing on the interval



concave upward
f' is increasing
$f'' > 0$



concave downward
f' is decreasing
$f'' < 0$

Test for Concavity

Let f be a function whose second derivative exists on ~~an~~ open interval I.

1) If $f''(x) > 0$ for all x in I, f is concave upward on I
2) If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I

e.g. $f(x) = \frac{6}{x^2+3}$

$$f'(x) = \frac{(x^2+3)(0) - 6(2x)}{(x^2+3)^2} = \frac{-12x}{(x^2+3)^2} = 0$$

Critical numbers: $x=0$

$$f''(x) = \frac{(x^2+3)^2(-12) - (-12x)[2(x^2+3)](2x)}{(x^2+3)^4}$$

$$= \frac{36(x^2-1)}{(x^2+3)^3} = 0$$

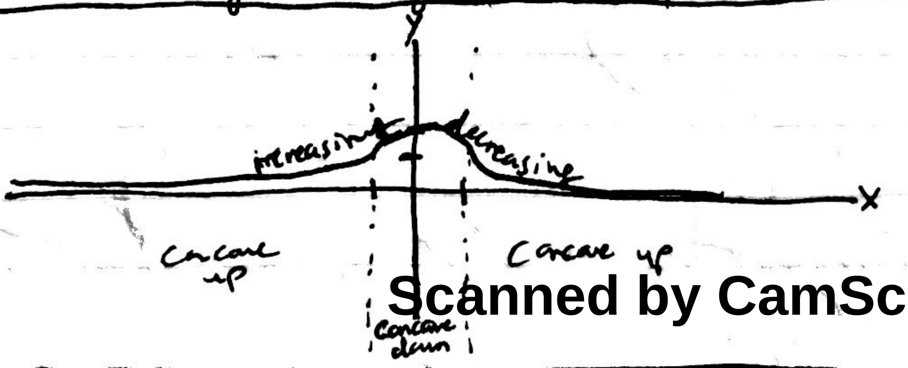
points of inflection: $x = 1, -1$

first derivative test:

Interval	$(-\infty, 0)$	$(0, \infty)$
test value	-1	1
$f'(x)$	+	−
Conclusion	increasing	decreasing

Second derivative test

Interval	$(-\infty, -1)$	$(-1, 1)$	$(1, \infty)$
test value	-2	0	2
$f''(x)$	+	−	+
Conclusion	concave up	concave down	concave up



Definition of point of inflection

If $(c, f(c))$ is a point of inflection of graph f, then $f''(c)=0$ or f'' DNE at $x=c$

A point of a inflection is where a graph changes from concave up to down or vice versa

Second Derivative Test

Let f be a function such that $f'(c)=0$ and the second derivative of f exists on an open interval containing c.

1) If $f''(c)>0$, then f has a relative min $(c, f(c))$
2) If $f''(c)<0$, then f has a relative maximum $(c, f(c))$

If $f''(c)=0$, the test fails; f may have a relative maximum, minimum, or neither

Definition of Limits at Infinity

Let L be a real number

1) The statement $\lim_{x\to\infty} f(x)=L$ means that for each $\varepsilon>0$, there exists an $M>0$ such that $|f(x)-L|<\varepsilon$ whenever $x>M$
2) That statement $\lim_{x\to-\infty} f(x)=L$ means that for each $\varepsilon>0$, there exists an $N<0$ such that $|f(x)-L|<\varepsilon$ whenever $x<N$

Limits at Infinity

If r is a positive rational number c is any real number, then $\lim_{x\to\infty} \frac{c}{x^r} = 0$

Furthermore, if x^r is defined when $x<0$, then

$$\lim_{x\to-\infty} \frac{c}{x^r} = 0$$

e.g. Find the limit

$$\lim_{x\to\infty} \left(5 - \frac{2}{x^2}\right)$$

$$\lim_{x\to\infty} 5 - \lim_{x\to\infty} \frac{2}{x^2}$$

$$5 - 0 = 5$$

e.g 2 Find the limit

$$\lim_{x\to\infty} \frac{2x+5}{3x^2+1}$$

$$\lim_{x\to\infty} \frac{\frac{2x}{x^2} + \frac{5}{x^2}}{\frac{3x^2}{x^2} + \frac{1}{x^2}}$$

$$\lim_{x\to\infty} \frac{\frac{2}{x} + \frac{5}{x^2}}{3 + \frac{1}{x^2}} = \frac{0+0}{3+0} =$$

$$\boxed{0}$$

e.g. 3 $\lim_{x\to\infty} \frac{2x^3+5}{3x^2+1}$

$$= \lim_{x\to\infty} \frac{\frac{2x^3}{x^3} + \frac{5}{x^3}}{\frac{3x^2}{x^3} + \frac{1}{x^3}}$$

$$= \frac{2+0}{0+0} = \frac{2}{0} =$$

$\boxed{\text{DNE}}$

Guidelines for finding limits at ±∞ of Rational Equations

1) If the degree for the numerator is less than the degree of the denominator, then the limit of the rational function is 0.
2) If the degree of the numerator is equal to the degree of the denominator, then the limit of the rational function is the ratio of the leading coefficients
3) If the degree of the numerator is greater than the degree of the denominator, then the limit of the rational function does not exist

eg $\lim_{x \to \infty} \frac{3x-2}{\sqrt{2x^2+1}} = \lim_{x \to \infty} \frac{\frac{3x}{x} - \frac{2}{x}}{\sqrt{\frac{2x^2}{x^2} + \frac{1}{x^2}}} = \frac{3 - \frac{2}{x}}{\sqrt{2 + \frac{1}{x^2}}} = \frac{3-0}{\sqrt{2+0}} = \frac{3}{\sqrt{2}}$

$\lim_{x \to \infty} \sin x = DNE$ $\qquad$ $\lim_{x \to \infty} \frac{\sin x}{x} = 0$

Guidelines for Analyzing the Graph of a Function

1. Determine the domain and range of the function.
2. Determine the intercepts, asymptotes, and symmetry of the graph.
3. Locate the x-values for which $f'(x)$ and $f''(x)$ are zero or DNE. Use results to determine relative extrema and points of inflection.

example $\quad f(x) = \dfrac{2(x^2-9)}{x^2-4} = \dfrac{2x^2-18}{x^2-4}$

$$f'(x) = \frac{20x}{(x^2-4)^2} = 0$$

$$x = 0$$

$$f''(x) = \frac{-20(3x^2+4)}{(x^2-4)^3} = 0$$

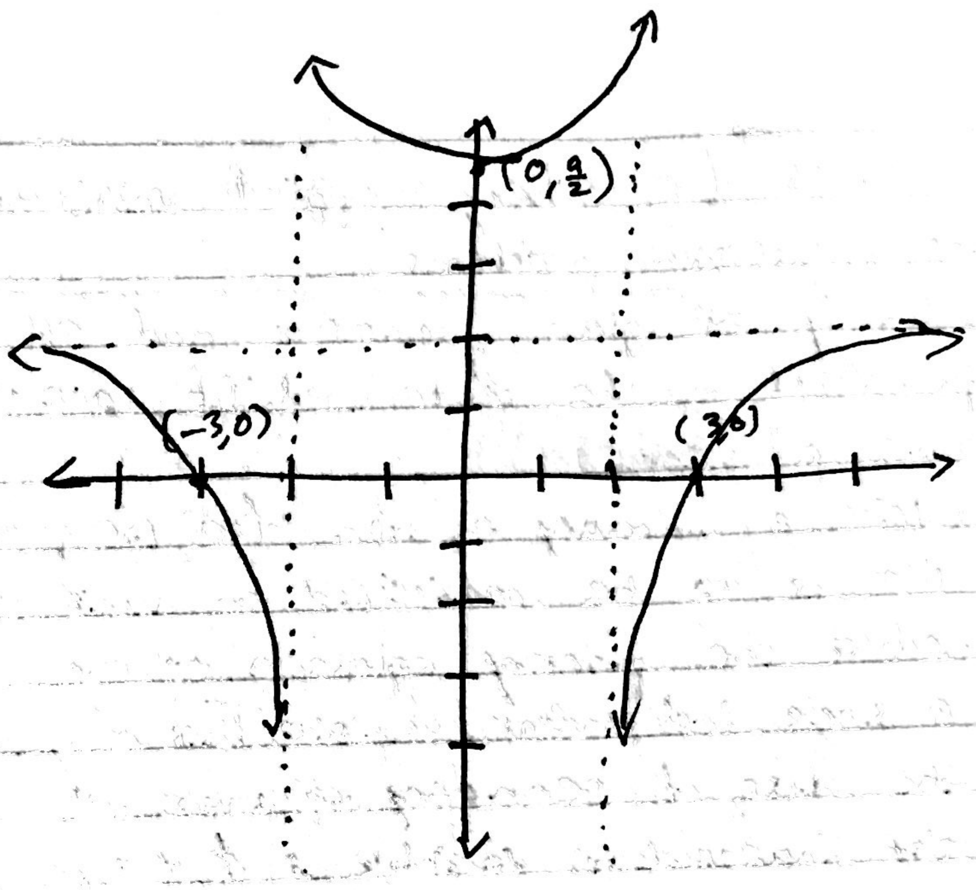
NO POI

X interapts: $(3,0)$ $(-3,0)$

y interapts: $(0, \frac{9}{2})$

Horizontal asymptote: $y = 2$

Vertical asymptote: $x = \pm 2$



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Guidelines for solving applied minimum and maximum problems

1) Identify all given quantities and all quantities to be determined. If possible, make a sketch
2) Write a primary equation for the quantity that is to be maximized or minimized.
3) Reduce the primary equation to one having a single independent variable. This may involve the use of secondary equation relating the independent variables of the primary equation
4) Determine the feasible domain of the primary equation. This is, determine the values for which the stated problem makes sense.
5) Determine the desired maximum or minimum value by the calculus techniques discussed in sections 3.1 through 3.4

e.g. A manufacturer wants to design an open box having a square base and a surface area of 108 in^2. What dimensions will produce a box with maximum volume?

Primary Function

$$V = x^2 h$$

Secondary Function

$s = $ (area of base) + (area of four sides)

$S = x^2 + 4xh = 108$

$$h = \frac{108 - x^2}{4x}$$

$$V = x^2\left(\frac{108 - x^2}{4x}\right)$$

$$= 27x - \frac{x^3}{4}$$

$$\frac{dv}{dx} = 27 - \frac{3x^2}{4} = 0$$

$$3x^2 = 108$$

$$x = \pm 6$$

$$x = 6, \; h = 3$$

Definition of Differentials

Let $y = f(x)$ represent a function that is differentiable on an open interval containing x. The differential of x (denoted by dx) is any nonzero real number.

The differential of y (denoted by dy) is

$$dy = f'(x)\,dx$$

e.g. $y=x^2$

Find dy when $x=1$ $dx=0.01$. Compare this value with Δy for $x=1$ & $\Delta x=0.01$

$dy = f'(x)dx$

$dy = 2x\,dx$

$dy = 2(1)(0.01)$

$= 0.02$

$\Delta y =$

$f(x+\Delta x) - f(x)$

$f(1.01) - f(1)$

$1.01^2 - 1^2$

$1.0201 - 1 = 0.0201$

Differential Functions

Let u and v be differentiable functions of x.

constant multiple: $d[cu] = c\,du$

sum or difference: $d[u \pm v] = du \pm dv$

product : $d[uv] = u\,dv + v\,du$

quotient : $d\left[\frac{u}{v}\right] = \frac{v\,du - u\,dv}{v^2}$

e.g. $y=x^2$

$\frac{dy}{dx} = 2x$ ← derivative

$dy = 2x\,dx$ ← differential

eg $y = 2\sin x$

$\frac{dy}{dx} = 2\cos x$

$dy = 2\cos x\,dx$